BRIESKORN SUBMANIFOLDS, LOCAL MOVES ON KNOTS, AND KNOT PRODUCTS

LOUIS H. KAUFFMAN AND EIJI OGASA

Abstract. We prove the following: Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let K and J be closed, oriented, $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifolds of S^{2p+3}. Then K is equivalent to J if and only if a Seifert matrix associated with a simple Seifert hypersurface for K is $(-1)^p$-S-equivalent to that for J. We also discuss the $2p+1=3$ case.

This result implies one of our main results: Let $\mu \in \mathbb{N}$. A 1-link A is pass-move equivalent to a 1-link B if and only if $A \otimes^\mu$ Hopf is $(2\mu+1, 2\mu+1)$-pass-move equivalent to $B \otimes^\mu$ Hopf.

It also implies the other of them: Two-fold cyclic suspension commutes with the performance of the twist move for spherical k-knots ($k \geqq 5$).

Furthermore it implies the following: Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let K be a closed oriented $(2p+1)$-dimensional submanifold of S^{2p+3}. Then K is a Brieskorn submanifold if and only if K is connected, $(p-1)$-connected, simple and has a $(p+1)$-Seifert matrix associated with a simple Seifert hypersurface that is $(-1)^p$-S-equivalent to a KN-type (see the body of the paper for a definition). We also discuss the $2p+1=3$ case.

1. Introduction

We begin this introduction with the two basic theorems that we prove in this paper. Theorem 6.3 tells us how the Seifert matrix for a simple codimension two submanifold of an odd-dimensional sphere determines the embedding of the submanifold. We then explain briefly, in this introduction, our applications of these results to knot products and local moves.

Theorem 6.3. (1) *Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let K and J be closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifolds of S^{2p+3}. Then* (i) *is equivalent to* (ii).

(i) *K is equivalent to J.*

E-mail: kauffman@uic.edu ogasa@mail1.meijigakuin.ac.jp

Keywords: Brieskorn manifolds, Brieskorn submanifolds, Seifert hypersurfaces, Seifert matrices, products of knots, local moves on 1-knots, local moves on high dimensional knots, crossing-changes on 1-links, pass-moves on 1-links, pass-moves on high dimensional links, twist-moves on high dimensional links.
MSC2000 57Q45, 57M25, 32S55.

(ii) *A simple Seifert matrix* P_K *for* K *is* $(-1)^p$-S*-equivalent to a simple Seifert matrix* P_J *for* J.

(2) *Let* K *and* J *be closed oriented 3-dimensional simple submanifolds of* S^5. *Then* (i) *is equivalent to* (ii).

(i) K *is equivalent to* J.

(ii) *There is a simple Seifert hypersurface* V_K *(resp.* V_J*) for* K *(resp.* J*) with the following properties: There is an orientation preserving diffeomorphism map* $f : V_K \to V_J$. V_K *(resp.* V_J*) consists of one 0-handle* h^0_K *(resp.* h^0_J*) and 2-handles* h^2_{Ki} *(resp.* h^2_{Jj}*).* $f(h^0_K) = h^0_J$. $f(h^2_{Kl}) = h^2_{Jl}$ *for each* l. $s(f(h^2_{Ka}), f(h^2_{Kb})) = s(h^2_{Ja}, h^2_{Jb})$ *for each pair* (a, b), *where* $s(h^2_{*a}, h^2_{*b})(* = K, J)$ *denotes a Seifert pairing of a pair of 2-cycles which are defined by* $h^2_{*\xi}(\xi = a, b)$.

Theorem 6.7(1) *Let* $2p + 1 \geq 5$ *and* $p \in \mathbb{N}$. *Let* K *be a closed oriented* $(2p + 1)$*-dimensional submanifold of* S^{2p+3}. *Then* (i) *is equivalent to* (ii).

(i) K *is a Brieskorn submanifold.*

(ii) K *is connected,* $(p - 1)$*-connected, simple, and has a simple Seifert matrix* P *that is* $(-1)^p$-S*-equivalent to a* KN*-type.*

(2) *Let* $a, b,$ *and* c *be integers* $\geqq 2$. *Let* J *be the Brieskorn submanifold* $\Sigma(a, b, c)$. *Let* K *be a closed oriented 3-dimensional submanifold of* S^5. *Then* (i) *is equivalent to* (ii).

(i) K *is equivalent to* J.

(ii) *The same as the condition of Theorem 6.3.*(2).(ii).

The terms and definitions needed for these theorems are in the body of the paper. (Theorem 6.3 implies Theorem 6.7.) The theorems are fundamental to the structure of simple submanifolds of spheres and have not been proven before. We apply these results to obtain proofs of our main new results, Theorems 4.2 and 4.3, about knot products and local moves on high dimensional knots. We leave it to the reader to find the exact statements of these theorems in §4 of this paper. These theorems allow us to make comparisons of the action of the tensor product (such as tensoring with a Hopf link) and certain local moves on high dimensional knots. In Theorem 4.2, we see that tensoring with the Hopf link commutes with the performance of the pass move. In Theorem 4.3, we see that two-fold cyclic suspension commutes with the performance of the twist move. Much of the work in this paper depends upon familiarity with knot products and local moves on knots. We take care to review these matters in the next three sections.

Table of Contents

2. Review of knot products

We work in the smooth category.

Let J (resp. K) be a (not necessarily spherical) closed oriented n-dimensional submanifold of S^{n+2} ($n \in \mathbb{N}$). (If J is homeomorphic to the standard sphere, J is said to be *spherical*.) If there is an orientation preserving diffeomorphism map $f : S^{n+2} \to S^{n+2}$ such that $f(J) = K$ and such that $f|_J$ is regarded as an orientation preserving diffeomorphism map $J \to K$, we say that J is equivalent to K. By obstruction theory there is a connected compact oriented $(n+1)$-dimensional submanifold W of S^{n+2} such that $\partial W = J$ (See e.g. Theorem 3 in P.50 of [14]). We call W a *Seifert hypersurface* for J. (Replace S^{n+2} with another (not necessarily closed) compact oriented $(n+2)$ manifold. If there is such W for J, we also call W a *Seifert hypersurface* for J.)

Suppose that a topological space A is a sub-topological space of a topological space B. Let $\overline{A}$ denote the closure of A in B. We do not say what B is if there is no danger of confusion.

Let M be a manifold-with-boundary. Let Int$M$ denote $M - \partial M$.

Let P be a (not necessarily closed) n-dimensional oriented submanifold of a (not necessarily closed) m-dimensional oriented manifold-with-boundary Q. Let $N(P)$ denote the tubular neighborhood of P in Q.

Let $f : C^n \longrightarrow C$ be a (complex) polynomial mapping with an isolated singularity at the origin of C^n. That is, $f(0) = 0$ and the complex gradient df has an isolated zero at the origin. The *link* of this singularity is defined by the formula $L(f) = V(f) \cap S^{2n-1}$. Here the symbol $V(f)$ denotes the variety of f, and S^{2n-1} is a sufficiently small sphere about the origin of C^n. Given another polynomial $g : C^m \longrightarrow C$, form $f + g$ with domain

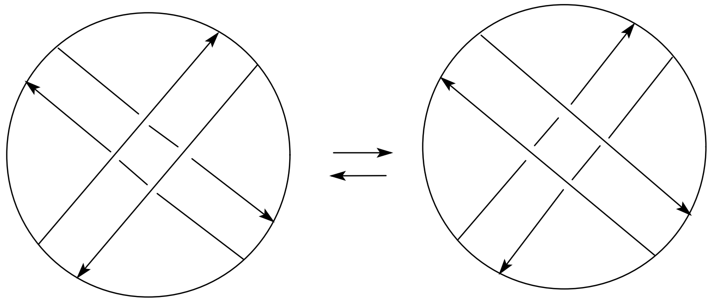


Figure 3.1: pass-move

$C^{n+m} = C^n \times C^m$ and consider $L(f+g) \subset S^{2n+2m-1}$. We use a topological construction for $L(f+g) \subset S^{2n+2m-1}$ in terms of $L(f) \subset S^{2n-1}$ and $L(g) \subset S^{2m-1}$. The construction generalizes the algebraic situation. Given nice codimension-two embeddings $K \subset S^{2n-1}$ and $L \subset S^{2m-1}$, we form a product $K \otimes L \subset S^{2n+2m-1}$. Then $L(f) \otimes L(-g) \simeq L(f+g)$. See [20].

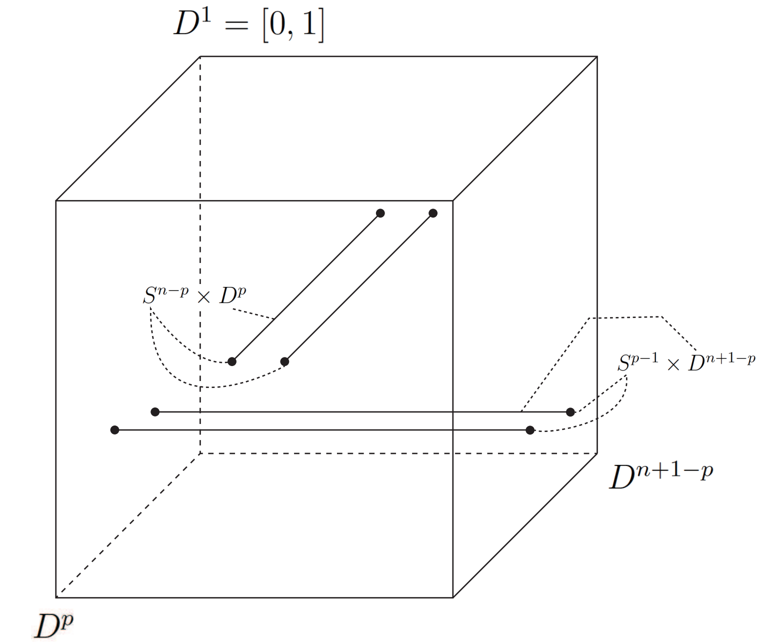
Let a and b be (not necessarily even) natural numbers≥ 2. Let $K \subset S^{a-1}$ and $L \subset S^{b-1}$ be (not necessarily connected,) codimension two, oriented, closed submanifolds. Take smooth maps $\begin{cases} f : D^a \to D^2 = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leqq 1\} \\ g : D^b \to D^2 \end{cases}$ such that $\begin{cases} f^{-1}((0,0)) \cap \partial D^a \\ g^{-1}((0,0)) \cap \partial D^b \end{cases}$ in $\begin{cases} \partial D^a \\ \partial D^b \end{cases}$ are $\begin{cases} K \subset S^{a-1} \\ L \subset S^{b-1}. \end{cases}$ Let $f+g$ be a smooth map

$$D^a \times D^b \to D^2$$
$$(x,y) \mapsto f(x) - g(y).$$

We define $K \otimes L$ to be $(f+g)^{-1}((0,0)) \cap \partial(D^a \times D^b)$ in $\partial(D^a \times D^b)$. If $S^{a-1} - K$ or $S^{b-1} - L$ is the total space of a S^1 fiber bundle such that each fiber is the interior of a Seifert hypersurface as in [8, 11], $K \otimes L$ is a smooth codimension two closed submanifold $\subset S^{a+b-1}$.

3. Review of local moves on high dimensional knots

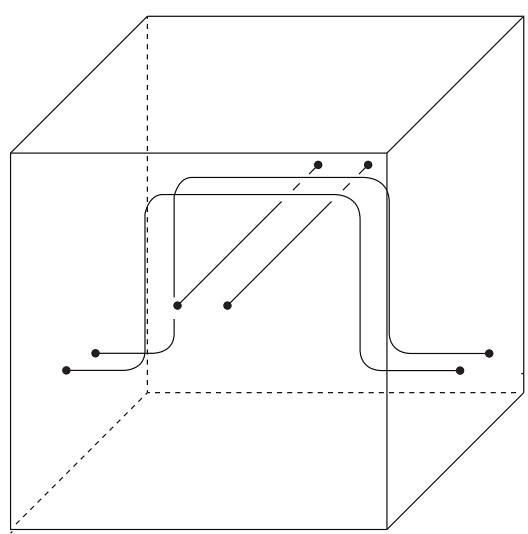
We review pass-moves on 1-links. See [9] for detail. Two 1-links are *pass-move-equivalent* if one is obtained from the other by a sequence of pass-moves. See Figure 3.1 for an illustration of the pass-move. Each of four arcs in the 3-ball may belong to different components of the 1-link. If K and J are pass-move-equivalent and if K and K' are equivalent, then we also say that K' and J are pass-move-equivalent.



This cube is $B = D^{n+2} = D^1 \times D^p \times D^{n+1-p}$

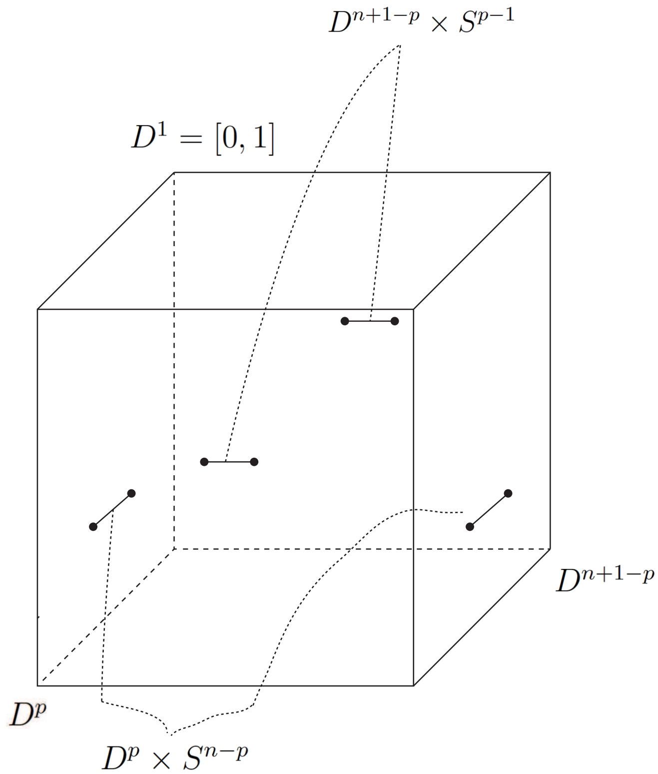
$B \cap K_+$

Figure 3.2.(1): A $(p, n+1-p)$-pass-move-triple



$B \cap K_-$

Figure 3.2.(2): A $(p, n+1-p)$-pass-move-triple



$B \cap K_0$

Figure 3.2.(3): A $(p, n+1-p)$**-pass-move-triple**

We review (p,q)-pass-moves on n-knots ($p, q \in \mathbb{N}$, $p+q = n+1$). Figure 3.2, which consists of the three figures (1), (2) and (3), is a diagram of a (p,q)-pass-move-triple. Confirm that, if $(p,q) = (1,1)$, the (p,q)-pass-move is the pass-move on 1-links.

Let K_+, K_-, K_0 be n-dimensional closed oriented submanifolds $\subset S^{n+2}$ ($n \in \mathbb{N}$). Let B be an $(n+2)$-ball trivially embedded in S^{n+2}. Suppose that K_+ coincides with K_-, K_0 in $\overline{S^{n+2} - B}$.

Take an $(n+1)$-dimensional p-handle $h^p_*(* = +, -)$ and an $(n+1)$-dimensional $(n+1-p)$-handle h^{n+1-p} in B with the following properties.

(1) $h^p_* \cap \partial B$ is the attaching part of h^p_*. $h^{n+1-p} \cap \partial B$ is the attaching part of h^{n+1-p}.

(2) h^p_* (resp. h^{n+1-p}) is embedded trivially in B.

(3) $$h^p_* \cap h^{n+1-p} = \phi.$$

(4) The attaching part of h^p_+ coincides with that of h^p_-. The linking number (in B) of

$$[h^p_+ \cup (-h^p_-)] \text{ and } [h^{n+1-p} \text{ whose attaching part is fixed in } \partial B]$$

is one if an orientation is given.

Let $K_*(*=+,-)$ satisfy that $K_* \cap \mathrm{Int}B = (\partial h^p_* - \partial B) \cup (\partial h^{n+1-p} - \partial B)$. Note the following. When we define K_+, h_+ exists in B and h_- does not exist in B. When we define K_-, h_- exists in B and h_+ does not exist in B.
Let

$$P = K_+ \cap (S^{n+2} - \mathrm{Int}B)$$

$$Q = h^p_+ \cap \partial B$$

$$R = h^{n+1-p} \cap \partial B$$

$$T = P \cup Q \cup R.$$

Then T is an n-dimensional oriented closed submanifold $\subset (S^{n+2} - \mathrm{Int}B) \subset S^{n+2}$. Let K_0 be $T \subset S^{n+2}$. Then we say that $(K_+,\ K_-,\ K_0)$ is related by a single $(p, n+1-p)$-*pass-move* in B. We also say that $(K_+,\ K_-,\ K_0)$ is a $(p, n+1-p)$-*pass-move-triple*. We say that K_+ and K_- differ by a single $(p, n+1-p)$-*pass-move* in B.

If $(K_+,\ K_-,\ K_0)$ is a $(p, n+1-p)$-pass-move-triple, then we also say that $(K_-,\ K_+,\ K_0)$ is a $(p, n+1-p)$-*pass-move-triple*. If K_+ and K_- differ by a single $(p, n+1-p)$-pass-move in B, then we also say that K_- and K_+ differ by a single $(p, n+1-p)$-*pass-move* in B.

Let (K_+, K_-, K_0) be related by a single $(p, n+1-p)$-pass-move in B. Then there is a Seifert hypersurface V_* for K_* $(* = +, -, 0)$ with the following properties.

(1) $$V_\sharp = V_0 \cup h^p_\sharp \cup h^{n+1-p} (\sharp = +, -).$$

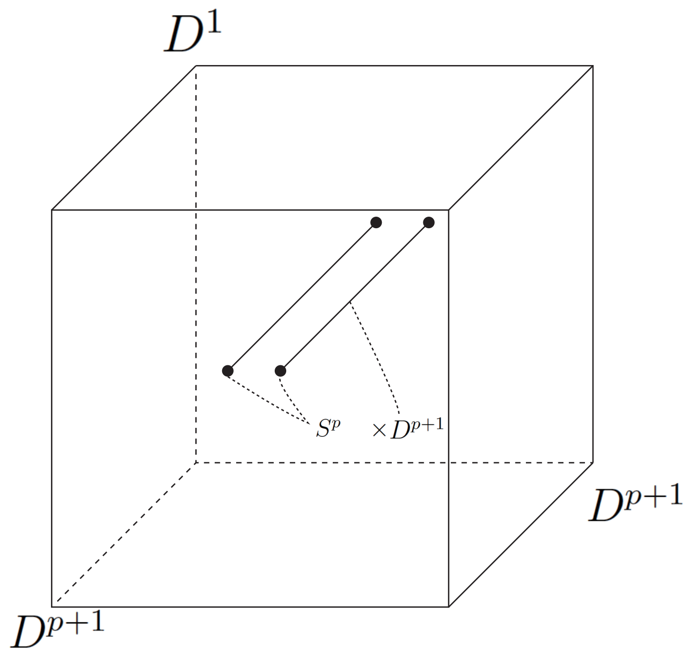
$$V_\sharp \cap B = h^p_\sharp \cup h^{n+1-p}.$$

(2) $$V_0 \cap \mathrm{Int}B = \phi.$$

$V_0 \cap \partial B$ is the attaching part of $h^p_\sharp \cup h^{n+1-p}$.
(The idea of the proof is the Thom-Pontrjagin construction.)

Then the ordered set (V_+, V_-, V_0) is called a $(p, n+1-p)$-*pass-move-triple of Seifert hypersurfaces* for (K_+, K_-, K_0). We say that an ordered set (V_+, V_-, V_0) is related by a single $(p, n+1-p)$-*pass-move* in B. We say that V_- (resp. V_+) is obtained from V_+ (resp. V_-) by a single $(p, n+1-p)$-*pass-move* in B.

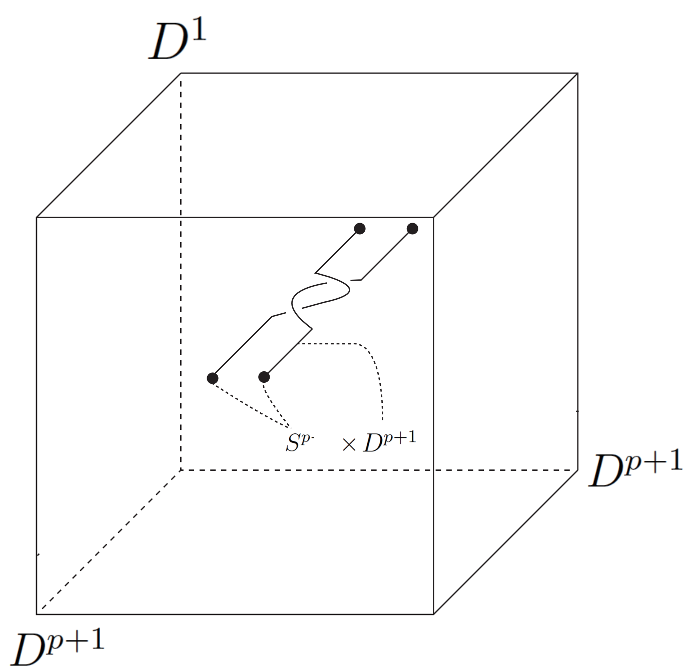
We review twist-moves on high dimensional knots. (Note: In [25] the twist-move is called the $XXII$-move.) Figure 3.3, which consists of the three figures (1), (2) and (3), is a diagram of a twist-move-triple. Confirm that if $p = 0$, the twist-move is the crossing change on 1-links.



This cube is $D^{2p+3} = B$.

$B \cap K_+$

Figure 3.3.(1): A twist-move-triple



$B \cap K_-$

Figure 3.3.(2): A twist-move-triple

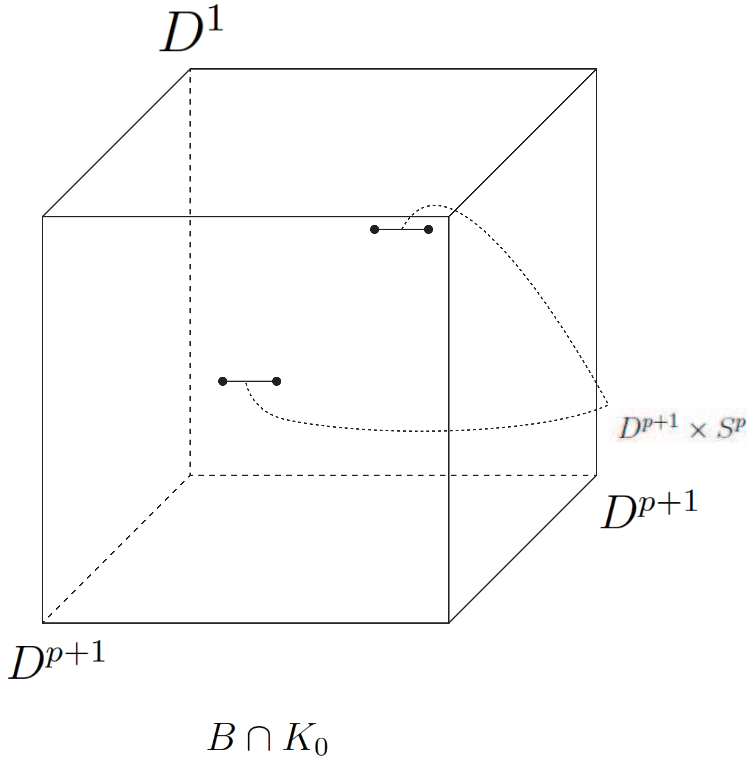


Figure 3.3.(3): **A twist-move-triple**

Let K_+, K_-, K_0 be $(2p+1)$-dimensional closed oriented submanifold $\subset S^{2p+3}$ ($p \in \mathbb{N} \cup \{0\}$). Let B be a $(2p+3)$-ball trivially embedded in S^{2p+3}. Suppose that K_+ coincides with K_-, K_0 in $\overline{S^{2p+3} - B}$. Take a single $(2p+2)$-dimensional $(p+1)$-handle h_+ (resp. h_-) embedded in B such that (the handle)$\cap \partial B$ is the attaching part of the handle.

Note. [5, 6, 34, 35, 36] etc. imply that the core of h_+ (resp. h_-) is trivially embedded in B under the above condition.

Suppose that (h_+- its attaching part)$\cap$($h_- -$ its attaching part)$= \phi$. Suppose that their attaching parts coincide. Thus we can suppose that we regard $h_+ \cup h_-$ as an oriented $(2p+2)$-submanifold $\subset S^{2p+3}$ if we give the opposite orientation to h_-. Then we can define a $(p+1)$-Seifert matrix for the $(2p+2)$-submanifold $h_+ \cup h_-$. We can suppose that the Seifert matrix is a 1×1-matrix (1).

Let $K_*(* = +, -)$ satisfy that $K_* \cap \mathrm{Int}B = (\partial h_* - \partial B)$. Note the following. When we define K_+, h_+ exists in B and h_- does not exist in B. When we define K_-, h_- exists in B and h_+ does not exist in B. Let $P = K_+ \cap (S^{2p+3} - \mathrm{Int}B)$. Let $Q = h_+ \cap \partial B$. Let $T = P \cup Q$. Then T is an $(2p+1)$-dimensional oriented closed submanifold in $S^{2p+3} - \mathrm{Int}B$. Let K_0 be T in S^{2p+3}. Then we say that an ordered set (K_+, K_-, K_0) is related by a single *twist-move*. (K_+, K_-, K_0) is called a *twist-move-triple*. We say that K_+ and K_- differ by a single *twist-move* in B. If (K_+, K_-, K_0) is a twist-move-triple, then we also say that (K_-, K_+, K_0) is a *twist-move-triple*. If K_+ and K_- differ by a single twist-move in B, we also say that K_- and K_+ differ by a single *twist-move* in B.

Note. Suppose that p is an odd natural number, put $p = 2k+1$. The twist-move for $(4k+3)$-submanifolds $\subset S^{4k+5}$ ($4k+3 \in \mathbb{N}$, $\quad k \in \mathbb{N} \cup \{0\}$) has the following property: Suppose that K_+ is made into K_- by the twist-move. Then K_- is a nonspherical knot in general even if K_+ is a spherical knot. Furthermore the $H_*(K_-;\mathbb{Z})$ is not congruent to $H_*(K_+;\mathbb{Z})$ in general. Example: A Seifert hypersurface V_* for a 3-knot K_* ($* = +, -$); Framed link representation of V_+ is the Hopf link such that the framing of one component is zero and that that of the other is two. Framed link representation of V_- is the Hopf link such that the framing of each component is two.

Let (K_+, K_-, K_0) be related by a single twist-move in B. Then there is a Seifert hypersurface V_* for K_* ($* = +, -, 0$) with the following properties.

(1) $V_\sharp = V_0 \cup h_\sharp$ ($\sharp = +, -$). $V_\sharp \cap B = h_\sharp$.

(2)
$$V_0 \cap \text{ Int } B = \phi.$$

$V_0 \cap \partial B$ is the attaching part of $h^p_\sharp$.

(The idea of the proof is the Thom-Pontrjagin construction.)

The ordered set (V_+, V_-, V_0) is called a *twist-move-triple of Seifert hypersurfaces* for (K_+, K_-, K_0). We say that V_- (resp. V_+) is obtained from V_+ (resp. V_-) by a single *twist-move* in B.

4. Main results and the background: Theorems on local moves on knots and products of knots

Knot products. In [8, 11] the following is proved: For any spherical n-knot K in S^{n+2} ($n \in \mathbb{N}$), take $K \otimes$Hopf, where we abbreviate $A \otimes$(the Hopf link) to $A \otimes$Hopf. Then we obtain a homomorphism $C_n \to C_{n+4}$, where C_n is the knot cobordism group in [17]. Furthermore it is the isomorphism if $n \in \mathbb{N} - \{1, 3\}$.

In [8, 11] the empty knot $[a]$ of degree a ($a \in \mathbb{N}$) was introduced. It is proved that any Brieskorn submanifold is equivalent to a $[a_1] \otimes ... \otimes [a_p]$ and that any $[a_1] \otimes ... \otimes [a_p]$ is equivalent to a Brieskorn submanifold.

Local moves on high dimensional knots. In [21] the following is proved: The $(p+1, p+1)$-pass-move on spherical $(2p+1)$-knots preserves the Arf invariant (resp. the signature) if p is even (resp. odd). Furthermore the following is proved: Let p be even (resp. odd). Simple $(2p+1)$-knots, K and J, are $(p+1, p+1)$-pass-move-equivalent if and only if their Arf invariant (resp. their signature) are the same.

In [23, 24, 25] the following are proved: The (1,2)-pass move (resp. the ribbon-move) on spherical 2-knots preserves the μ-invariant of 2-knots, the $\mathbb{Q}/\mathbb{Z}$-valued $\widetilde{\eta}$-invariants of 2-knots, the Farber-Levine pairing of 2-knots, and partial information of the cup product of three elements in H^1_{cpt}(the complement of each 2-knot). Note that the ribbon-move is a kind of local moves on 2-links.

In [26] the following is proved: For the Alexander polynomial $A(\ )$ of high dimensional knots we have an identity

$$A(K_+) - A(K_-) = (t-1)A(K_0)$$

associated with the twist move on $(4k+1)$-dimensional knots, where $k \in \mathbb{N} \cup \{0\}$ (resp. the (p,q)-pass-move, where $p \neq q$, on high dimensional knots).

Main results and the background. In [12, 13] we combined research of knot products and that of local moves on high dimensional knots, and obtained many results. We cite main results of them below. We generalize them and obtain our main new results, Theorems 4.2 and 4.3, of this paper.

We prove the following Theorems 4.1 and 4.2 on high dimensional pass-moves.

In Theorem 8.1 of [12] we proved the following: If a 1-knot A is obtained from a 1-knot B by one pass-move, then $A \otimes^\mu \mathrm{Hopf}$ is obtained from $B \otimes^\mu \mathrm{Hopf}$ by one $(2\mu+1, 2\mu+1)$-pass-move.

In Theorem 8.10 of [12] we proved the case where A is a knot in Theorem 4.1.

Theorem 4.1. *Let A be a 1-link. Let $\mu \in \mathbb{N}$. Let $J = A \otimes^\mu \mathrm{Hopf}$. Let K be obtained from J by one $(2\mu+1, 2\mu+1)$-pass-move. Then there is a 1-link B such that $K = B \otimes^\mu \mathrm{Hopf}$ and such that A is pass-move equivalent to B.*

Theorems 8.1 and 8.10 of [12] imply the following: A 1-knot A is pass-move equivalent to a 1-knot B if and only if $A \otimes^\mu \mathrm{Hopf}$ is $(2\mu+1, 2\mu+1)$-pass-move equivalent to $B \otimes^\mu \mathrm{Hopf}$. We generalize this to the link case as follows.

In Theorem 4.1 of [13] we proved the following: If a 1-link A is obtained from a 1-link B by one pass-move, then $A \otimes^\mu \mathrm{Hopf}$ is obtained from $B \otimes^\mu \mathrm{Hopf}$ by one $(2\mu+1, 2\mu+1)$-pass-move (see Note 12.5: This fact is also proved by using Theorem 6.3). By this fact and Theorem 4.1 we have the following.

Theorem 4.2. *Let $\mu \in \mathbb{N}$. A 1-link A is pass-move equivalent to a 1-link B if and only if $A \otimes^\mu \mathrm{Hopf}$ is $(2\mu+1, 2\mu+1)$-pass-move equivalent to $B \otimes^\mu \mathrm{Hopf}$.*

We prove the following Theorem 4.3 on twist-moves on high dimensional knots.

Theorem 4.1 of [12] proved the following: Suppose that two 1-links J and K differ by a single crossing-change. Then the knot products, $J \otimes^\mu \mathrm{Hopf}$ and $K \otimes^\mu \mathrm{Hopf}$, differ by a single twist-move, where $\mu \in \mathbb{N} \cup \{0\}$.

Theorem 7.1 of [12] proved the following: Let $m \in \mathbb{N} \cup \{0\}$. Suppose that two (not necessarily connected) $(2m+1)$-dimensional closed oriented submanifolds $\subset S^{2m+3}$, J and K, differ by a single twist-move. Then the $(2m+2\nu+1)$-submanifolds $\subset S^{2m+2\nu+3}$, $J \otimes^\nu [2]$ and $K \otimes^\nu [2]$, differ by a single twist-move. Note that Hopf$= [2] \otimes [2]$

Theorem 7.3 of [12] proved the following: Let $k \in \mathbb{N}$. Let K (resp. J) be $(4k+5)$-dimensional smooth submanifold $\subset S^{4k+7}$. Suppose that K and J differ by a single twist-move and are nonequivalent. Suppose that K is equivalent to $A \otimes^{k+1}$ Hopf for a 1-knot A. Then there is a unique equivalence class of simple $(4k+1)$-knots for K (resp. J) with the following properties.

(i) There is a representative element K' of the above equivalence class for K such that K is equivalent to $K' \otimes$ Hopf.

(ii) There is a representative element J' of the above equivalence class for J such that J is equivalent to $J' \otimes$ Hopf.

(iii) K' and J' differ by a single twist-move and are nonequivalent.

Compare the above Theorem 7.3 of [12] with the following Theorem 4.3.

Theorem 4.3. *Let $2p+1 \geq 7$ and $p \in \mathbb{N}$. Let J be a $(2p+1)$-dimensional smooth submanifold $\subset S^{2p+3}$. Suppose that J and K differ by a single twist-move and are nonequivalent. Suppose that J is equivalent to $A \otimes [2]$ for a $(2p-1)$-dimensional connected, $(p-2)$-connected, simple submanifold $A \subset S^{2p+1}$. Then there is a $(2p-1)$-dimensional connected, $(p-2)$-connected, simple submanifold $B \subset S^{2p+1}$ with the following properties:*

(i) *K is equivalent to $B \otimes [2]$.*

(ii) *A and B differ by a single twist-move and are nonequivalent.*

(iii) *The equivalence class of such B is unique.*

In order to prove the above results in [12, 13], it is enough to use [15, 18]. However in order to generalize the above results in [12, 13] and to obtain Theorems 4.1, 4.2, and 4.3, we need to prove Theorem 6.3 which are generalizations of [15, 18]. Theorem 6.3 implies Theorem 6.7, which is a result on Brieskorn submanifolds.

In [12, 13] we furthermore proved the following: For any simple n-knot K in S^{n+2} ($n \in \mathbb{N}$), take $K \otimes$Hopf. Then we obtain a map $\mathcal{S}_n \to \mathcal{S}_{n+4}$, where $\mathcal{S}_n$ is a set of simple n-knots. (See §5 for simple knots.) Furthermore it is the bijective map if $n = 5$ or $n \geqq 7$.

In [12] the following is proved: For the Alexander polynomial $A(\ )$ of $(4k+3)$-dimensional knots ($k \in \mathbb{N} \cup \{0\}$), we have an identity

$$A(K_+) - A(K_-) = (t+1)A(K_0)$$

associated with the twist move. It is a new type of local move identities of knot polynomial.

5. Review of simple submanifolds and Brieskorn submanifolds

In order to prove our main results, Theorems 4.2 and 4.3, we need to prove Theorem 6.3 in §6. In order to state Theorem 6.3 we review the following.

Let K be a (not necessarily spherical) connected, closed, oriented, $(2p+1)$-dimensional submanifold of S^{2p+3} ($p \in \mathbb{N}$). K is said to be *simple* if K satisfies that $\pi_1(S^{2p+3} - N(K)) \cong \mathbb{Z}$ and $\pi_i(S^{2p+3} - N(K)) \cong 0$ ($2 \leqq i \leqq p$). Let V be a Seifert hypersurface for K. V is said to be *simple* if $\pi_i(V)$ is trivial ($1 \leqq i \leqq p$). See Theorem 7.1.

Let V be a Seifert hypersurface for a closed oriented $(2p+1)$-dimensional submanifold $K \subset S^{n+2}$. Let $x_1, ..., x_\mu$ be $(p+1)$-cycles in V which compose an ordered basis of $H_{p+1}(V;\mathbb{Z})/\mathrm{Tor}$. Recall that the orientation of V is compatible with that of K. Push x_i in the positive direction of the normal bundle of V. Call it x_i^+. Push x_i in the negative direction of the normal bundle of V. Call it x_i^-. A $(p+1)$-(*positive*) *Seifert matrix* for K associated with V represented by the ordered basis, $\{x_1, ..., x_\mu\}$, is a $(\mu \times \mu)$-matrix

$$S = (s_{ij}) = (\mathrm{lk}(x_i, x_j^+)).$$

A $(p+1)$-*negative Seifert matrix* for K associated with V represented by the ordered basis, $\{x_1, ..., x_\mu\}$, is a $(\mu \times \mu)$-matrix

$$N = (n_{ij}) = (\mathrm{lk}(x_i, x_j^-)).$$

We sometimes omit to write K, V, and $\{x_i\}$ when they are clear from the context. See e.g. [12] for $(p, n+1-p)$-Seifert matrices for n-knots ($p, n \in \mathbb{N}$).

Let A be an $r \times r$-matrix. Let P be a unimodular $r \times r$-matrix. We say that A is *equivalent* to A' if $A' = {}^tPAP$, where ^{t} denotes an operation of making a transposed matrix.

Proposition 5.1. (**Well-known**.) *Let S (resp. S') be a $(p+1)$-positive Seifert matrix for the above K associated with the above V represented by an ordered basis $\{x_i\}$ (resp. $\{x'_i\}$) of $(p+1)$-cycles. Then S is equivalent to S'.*

If a $(p+1)$-positive Seifert matrix P for K and a $(p+1)$-negative Seifert matrix N for K are defined by the same ordered basis $\{x_i\}$, we say that P and N are *related* and that the pair (P, N) is a *pair of $(p+1)$-related Seifert matrices* for K.

Proposition 5.2. (**Well-known**.) *Let X be a $(p+1)$-positive Seifert matrix for a $(2p+1)$-dimensional closed oriented submanifold $K \subset S^{2p+3}$ associated with a Seifert hypersurface V. Then $(-1)^p\, {}^tX$ represents the $(p+1)$-negative Seifert matrix related to X for K. Furthermore $X - (-1)^p\, {}^tX$ represents the intersection product on $H_{p+1}(V;\mathbb{Z})/\mathrm{Tor}$.*

Let A be an $r \times r$-matrix ($r \in \mathbb{N} \cup \{0\}$). We say that A is *$(-1)^p$-S-equivalent* to
$\begin{pmatrix} A & \alpha & 0 \\ (-1)^p\, {}^t\alpha & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} A & \beta & 0 \\ (-1)^p\, {}^t\beta & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$),
where α and β are column vectors. We say that A is *$(-1)^p$-S-equivalent* to A' if A is

equivalent to A'. We say that A is $(-1)^p$-*S-equivalent* to C if A is $(-1)^p$-S-equivalent to B and if B is $(-1)^p$-S-equivalent to C.

Let K be a closed oriented $(2p+1)$-dimensional submanifold of S^{2p+3} ($p \in \mathbb{N}$). If A is a $(p+1)$-Seifert matrix (resp. negative $(p+1)$-Seifert matrix) associated with a simple Seifert hypersurface V for K, we call A a *simple Seifert matrix* (resp. *negative simple Seifert matrix*) for K. We can define a *pair of related simple Seifert matrices* for K.

Note. If we say that K has a simple Seifert matrix A, then it means that K has a simple Seifert hypersurface.

Let a_* be an integer$\geqq 2 (* = 1, ..., q)$, where $q \in \mathbb{N}$. The submanifold $\{(z_1, ..., z_q) | |z_1|^2 +,,, + |z_q|^2 = 1, z_* \in \mathbb{C}\} \cap \{(z_1, ..., z_q) | z_1^{a_1} + ... + z_q^{a_q} = 0, z_* \in \mathbb{C}\}$ $\subset \{(z_1, ..., z_q) | |z_1|^2 +,,, + |z_q|^2 = 1, z_* \in \mathbb{C}\}$ is called the *Brieskorn submanifold* $\Sigma(a_1, ..., a_q)$. The oriented diffeomorphism type of the Brieskorn submanifold $\Sigma(a_1, ..., a_q)$ is called the *Brieskorn manifold* $\Sigma(a_1, ..., a_q)$.

Let a be an integer$\geqq 2$. Let $\Lambda_a = (\zeta_{i,j})$ be an $(a-1) \times (a-1)$ matrix such that

$$(\zeta_{i,j}) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } j = i+1 \\ 0 & \text{else,} \end{cases} \qquad \text{that is,} \qquad (\zeta_{i,j}) = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \cdot & \cdot & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}.$$

Let $\Lambda_{a_1,...,a_q} = (-1)^{\frac{(q-1)q}{2}} \Lambda_{a_1} \otimes ... \otimes \Lambda_{a_q}$. It is called a *Kauffman-Neumann-type*, or a *KN-type*. See the last few paragraphs of §6 in [11] for KN-types.

The Brieskorn submanifold $\Sigma(a, b) \subset S^3$ is the torus (a, b) knot (See [20]). We say that the Brieskorn submanifold $\Sigma(a) \subset S^1$ is the empty knot of degree a (See [11]).

Theorem 5.3. ([8, 11].) *Let a_* be an integer$\geqq 2 (* = 1, ..., q)$. Let $q \geqq 3$. Any Brieskorn submanifold $\Sigma(a_1, ..., a_q)$ is a (not necessarily spherical) $(2q-3)$-dimensional connected, $(q-3)$-connected, simple submanifold of S^{2q-1} such that $\Lambda_{a_1,...,a_q}$ is a simple Seifert matrix.*

We prove that the converse of the $q \geqq 4$ case of Theorem 5.3 holds and that that of the $q = 3$ case does not hold in general. See Theorem 6.3, Note 6.6 and Theorem 6.7.

6. Theorems on simple submanifolds and Brieskorn submanifolds

Theorem 6.1. *Let $p \in \mathbb{N}$. Let K be a closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold of S^{2p+3}. Let P be a simple Seifert matrix for K. Then the following two conditions are equivalent.*

(i) *P' is a simple Seifert matrix for K.*

(ii) *P' is $(-1)^p$-S-equivalent to P.*

Note that we have the following proposition.

Proposition 6.2. *There is a natural number p and a closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold K of S^{2p+3} with the following property: There is a simple Seifert matrix P and a Seifert matrix R for K such that P is not $(-1)^p$-S-equivalent to R.*

Note. In Proposition 6.2, R is not a simple Seifert matrix. If R is associated with a Seifert hypersurface W for K, then W is not a simple Seifert hypersurface.

In [18] it was proved that if K and J are spherical, the following Theorem 6.3 is true. We generalize the result in [18] and prove a stronger theorem, which is Theorem 6.3.

Theorem 6.3. (1) *Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let K and J be closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifolds of S^{2p+3}. Then* (i) *is equivalent to* (ii)*.*

(i) *K is equivalent to J.*

(ii) *A simple Seifert matrix P_K for K is $(-1)^p$-S-equivalent to a simple Seifert matrix P_J for J.*

(2) *Let K and J be closed oriented 3-dimensional simple submanifolds of S^5. Then* (i) *is equivalent to* (ii)*.*

(i) *K is equivalent to J.*

(ii) *There is a simple Seifert hypersurface V_K (resp. V_J) for K (resp. J) with the following properties: There is an orientation preserving diffeomorphism map $f : V_K \to V_J$. V_K (resp. V_J) consists of one 0-handle h^0_K (resp. h^0_J) and 2-handles h^2_{Ki} (resp. h^2_{Jj}). $f(h^0_K) = h^0_J$. $f(h^2_{Kl}) = h^2_{Jl}$ for each l. $s(f(h^2_{Ka}), f(h^2_{Kb})) = s(h^2_{Ja}, h^2_{Jb})$ for each pair (a, b), where $s(h^2_{*a}, h^2_{*b})(* = K, J)$ denotes a Seifert pairing of a pair of 2-cycles which are defined by $h^2_{*\xi}(\xi = a, b)$.*

Note 6.4. The results in the first page of [3] and the results in the first page of [7] claim the following: Add the condition 'K and J are fibered knots' to Theorem 6.3.(1). Then (i) and (ii) are equivalent under this condition.

On the other hand, in Theorem 6.3.(1), we do not suppose that K is fibered nor that J is fibered. Therefore our result, Theorem 6.3.(1), is stronger than the results in [3, 7].

Note 6.5. Theorem 3.1 of [29] claims the following: There is a pair (K, J) with the following property: K and J are closed oriented 3-dimensional simple submanifolds of S^5. K is diffeomorphic to J. K is nonequivalent to J. There is a simple Seifert matrix P_K (resp. P_J) for K (resp. J) such that P_K is (-1)-S-equivalent to P_J.

Theorem 2.2 of [29] claims the following: Add the condition 'K and J are fibered knots' to Theorem 6.3.(2). Then (i) and (ii) are equivalent under this condition.

On the other hand, in Theorem 6.3.(2), we do not suppose that K is fibered nor that J is fibered. Therefore our result, Theorem 6.3.(2), is stronger than Theorem 2.2 of [29].

Theorem 2.2 of [28] claims the following without a proof: Add the condition 'K and J are diffeomorphic to a homology 3-sphere' to Theorem 6.3.(2). Then (i) and (ii) are equivalent under this condition.

On the other hand, in Theorem 6.3.(2), we do not suppose that K is diffeomorphic to a homology 3-sphere nor that J is diffeomorphic to a homology 3-sphere. Therefore our result, Theorem 6.3.(2), is stronger than Theorem 2.2 of [28].

Note. Theorem 6.3.(2) is not trivial. Reason: If we assume a similar condition for 1-knots (resp. 1-links) to Theorem 6.3.(2).(ii), that is, replace 2-handles (resp. 2-cycles) with 1-handles (resp. 1-cycles), then the 1-knots (resp. 1-links), K and J, which we obtain, are not equivalent in general.

By using [31, 32], we could omit the conditions on handles in Theorem 6.3.(2).(ii) and sophisticate it.

Note 6.6. On Theorem 6.3.(2) and the paragraph under Theorem 5.3, note the following. There is a closed oriented simple 3-dimensional submanifold $E \subset S^5$ with the following properties:

(i) There is a simple Seifert matrix P for E which is (-1)-S-equivalent to the empty matrix.

(ii) E is not homeomorphic to the 3-sphere.

See Note to Proof of Theorem 7.2.(1) in [12].

By Theorem 6.3 the converse of the $q \geqq 4$ case of Theorem 5.3 holds. See the following Theorem 6.7.(1). Note $q = p + 2$

Theorem 6.7. (1) *Let* $2p + 1 \geq 5$ *and* $p \in \mathbb{N}$. *Let* K *be a closed oriented* $(2p + 1)$*-dimensional submanifold of* S^{2p+3}. *Then* (i) *is equivalent to* (ii).

(i) K *is a Brieskorn submanifold.*

(ii) K *is connected,* $(p - 1)$*-connected, simple, and has a simple Seifert matrix* P *that is* $(-1)^p$*-*S*-equivalent to a* KN*-type.*

(2) *Let* $a, b,$ *and* c *be integers* $\geqq 2$. *Let* J *be the Brieskorn submanifold* $\Sigma(a, b, c)$. *Let* K *be a closed oriented 3-dimensional submanifold of* S^5. *Then* (i) *is equivalent to* (ii).

(i) K *is equivalent to* J.

(ii) *The same as the condition of Theorem 6.3.*(2).(ii).

Note 6.8. On Theorem 6.7.(1), we have a similar situation to Note 6.4 as follows. The results in the first page of [3] and the results in the first page of [7] claim the following:

Add the condition 'K and J are fibered knots' to Theorem 6.7.(1). Then (i) and (ii) are equivalent under this condition.

On the other hand, in Theorem 6.7.(1), we do not suppose that K is fibered nor that J is fibered. Therefore our result, Theorem 6.7.(1), is stronger than the results in [3, 7].

By Theorem 6.3 we have the following.

Theorem 6.9. (1) *Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let K be a closed oriented $(2p+1)$-dimensional submanifold of S^{2p+3}. Let A be a 1-link. Let a_* be an integer ≥ 2 $(* = 1, ..., p)$. Then* (i) *is equivalent to* (ii).
(i) $K = A \otimes [a_1] \otimes ... \otimes [a_p]$.
(ii) *K is connected, $(p-1)$-connected, and simple. There is a simple Seifert matrix P for K and a Seifert matrix P' for A such that P is $(-1)^p$-S-equivalent to $P' \otimes \Lambda_{a_1,...,a_p}$.*

(2) *Let $2p+1 \geq 5$ and $p \in \mathbb{N}$. Let K be a closed oriented $(2p+1)$-dimensional submanifold of S^{2p+3}. Let $p > q$. Let $2q+1 \geq 3$ and $q \in \mathbb{N}$. Let A be a simple $(2q+1)$-submanifold. Let a_* be an integer ≥ 2 $(* = 1, ..., p-q)$. Then* (i) *is equivalent to* (ii).
(i) $K = A \otimes [a_1] \otimes ... \otimes [a_{p-q}]$.
(ii) *K is connected, $(p-1)$-connected, and simple. There is a simple Seifert matrix P for K and a simple Seifert matrix P' for A such that P is $(-1)^p$-S-equivalent to*
$(-1)^{q(p-q)} P' \otimes \Lambda_{a_1,...,a_{p-q}}$.

Note 6.10. In Theorem 6.9 we use the following fact, which is proved in §6 of [11]: Let X (resp. Y) be a $(2x+1)$-(resp. $(2y+1)$-)dimensional closed oriented simple submanifold $\subset S^{2x+3}$ (resp. S^{2y+3}), where $x, y \in \mathbb{N} \cup \{-1, 0\}$. Here, we regard X (resp. Y) as a 1-link if x (resp. y) is 0, and as the empty knot if x (resp. y) is -1. Let S_X (resp. S_Y) be an $(x+1)$-(resp. $(y+1)$-)positive Seifert matrix for X (resp. Y). Let
$S_{X \otimes Y} = (-1)^{xy} S_X \otimes S_Y$.
Then $S_{X \otimes Y}$ is an $(x+y+3)$-positive Seifert matrix for $X \otimes Y$. Furthermore it holds that $X \otimes Y$ is a $(2x+2y+5)$-dimensional closed oriented simple submanifold $\subset S^{2x+2y+7}$ if $2x+2y+5 \geqq 3$.

Let M be a closed oriented m-dimensional manifold which we can embed in S^{m+2}. Let K be a closed oriented m-dimensional submanifold$\subset S^{m+2}$ which is diffeomorphic to M. Take a tubular neighborhood of K in S^{m+2}. Note that it is diffeomorphic to $M \times D^2$. Of course $\partial(M \times D^2) = M \times \partial D^2 = M \times S^1$.

Question 6.11. Let $f : M \times S^1 \to M \times S^1$ be a diffeomorphism map which is represented by (g, θ), where g is a diffeomorphism map of M and $\theta \in S^1$. Attach $M \times D^2$ with $\overline{S^{m+2} - N(K)}$ by such a diffeomorphism f. Suppose that $(M \times D^2) \cup_f \overline{S^{m+2} - N(K)}$ is diffeomorphic to S^{m+2}. Thus we obtain a submanifold $M \times \{0\} \subset S^{m+2}$. Call this submanifold K_f. Then is K_f equivalent to K?

Question 6.12. In Question 6.11, replace f with any diffeomorphism of $M \times S^1$. What is the answer?

In Question 6.11 (resp. Question 6.12), if M is the standard sphere, there is an f such that K_f is nonequivalent to K by [2]. We have the positive answer to Question 6.12 under some conditions on K if M is the standard sphere by [4, 18]. Theorem 4.6 of [29] claims that there are some nonspherical manifolds M which give the negative answer to Question 6.12.

We have a partial solution to Question 6.11 by Theorem 6.3.(1). That is, we have the following:

Corollary 6.13. *Let $2p+1 \geqq 5$ and $p \in \mathbb{N}$. In Question 6.11, if K is a closed oriented $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold of S^{2p+3}, we have the positive answer.*

Note. In [18] it was proved that if K is spherical, Corollary 6.13 is true. Corollary 6.13 is its generalization.

7. Handle decompositions of Seifert hypersurfaces

We use Theorem 7.1 in order to prove our other theorems in this paper. A special case of Theorem 7.1 is proved in [15, 18], which includes the case where K is PL homeomorphic to the standard sphere. Theorem 7.1 is stronger than this special case.

We review handle decompositions. See [1, 19, 30, 33] for detail.

Let W be a w-dimensional compact manifold ($w \in \mathbb{N}$). Take a handle decomposition $(B\times[0,1])\cup$(0-handles)$\cup ... \cup$(w-handles)$\cup(T\times[0,1])$, where there may not be an i-handle $(0 \leqq i \leqq w)$. Note that B (resp. T) is a compact $(w-1)$-dimensional submanifold of ∂W. Note that ∂B is diffeomorphic to ∂T. Note that ∂W is diffeomorphic to $B \cup_\alpha T$, where α is a diffeomorphism map from $\partial B \to \partial T$. If a handle is attached to $B \times [0,1]$, its attaching part is embedded in $B \times \{1\}$. No handle is attached to $T \times [0,1]$ although (a handle)$\cap(T \times [0,1]) \neq \phi$ may hold. B (resp. T) may be the empty set. We do not suppose whether B (resp. T) is connected or not. We do not suppose whether B (resp. T) is closed or not. We say that B is the *bottom* of this handle decomposition and that T is the *top* of this handle decomposition.

For a handle decomposition, regard the core (resp. cocore) of each handle as the cocore (resp. core) of it and replace the top (resp. the bottom) with the bottom (resp. the top), then we obtain a new handle decomposition. It is called the *dual handle decomposition* of the handle decomposition. If a handle h in a handle decomposition is changed into a handle $\bar{h}$ in its dual handle decomposition, $\bar{h}$ is called the dual handle of h. Note that we have the following: If a dual handle is attached to $T\times[0,1]$, its attaching part is embedded in $T \times \{0\}$. No dual handle is attached to $B \times [0,1]$ although (a handle)$\cap(B \times [0,1]) \neq \phi$ may hold.

Let V be a compact $(n+1)$-dimensional manifold $(n+1 \in \mathbb{N})$. For the convenience of the application (see Theorem 7.1), we suppose that the dimension is $n+1$. If a handle decomposition of V satisfies the following conditions, we say that the handle decomposition is a *special handle decomposition* of V.

(1) The top T is connected or empty. The bottom B is connected or empty.

(2) It has only one (resp. no) $(n+1)$-dimensional 0-handle if $B = \phi$ (resp. $B \neq \phi$). It has no $(n+1)$-dimensional i-handle $(1 \leqq i \leqq [\frac{n-1}{2}])$.

(3) The dual handle decomposition has only one (resp. no) $(n+1)$-dimensional 0-handle if $T = \phi$ (resp. $T \neq \phi$). The dual handle decomposition has no $(n+1)$-dimensional i-handle $(1 \leqq i \leqq [\frac{n-1}{2}])$.

Example. If the above V is 6-dimensional and has a special decomposition with $B = \phi$ and $T = \partial V$, then V has a handle decomposition (one 0-handle)$\cup$(3-handles), where there may be no 3-handle. If the above V is 7-dimensional and has a special decomposition $B = \phi$ and $T = \partial V$, then V has a handle decomposition (one 0-handle)$\cup$(3-handles)$\cup$(4-handles), where there may be no 3-handle, or where there may be no 4-handle.

We define 'surgeries by using embedded handles' as follows: Let X be an x-dimensional 'submanifold-with-boundary' of an m-dimensional 'manifold-with-boundary' M $(x, m \in \mathbb{N}, x < m)$. Suppose that we can embed $X \times [0,1]$ in M so that $X \times \{0\} = X$. Suppose that an $(x+1)$-dimensional p-handle h^p is embedded in M and is attached to $X \times [0,1]$ $(p \in \mathbb{N} \cup \{0\}, 0 \leqq p \leqq x)$. Suppose that the attaching part of h^p is embedded in $X \times \{1\}$. Suppose that $h^p \cap (X \times [0,1])$ is only the attaching part of h^p. Let $X' = \overline{\partial(h^p \cup (X \times [0,1])) - (X \times \{0\})}$. Note that there are two cases, $\partial X = \phi$ and $\partial X \neq \phi$. Note that X is the bottom and X' the top of this handle decomposition. Then we say that X' is obtained from X by *the surgery by using the embedded handle* h^p. We do not say that we use $X \times [0,1]$ if there is no danger of confusion when we use surgeries by using embedded handles.

Let $n \in \mathbb{N} \cup \{0\}$. Let K be an n-dimensional oriented closed submanifold of a (not necessarily closed) $(n+2)$-dimensional oriented compact manifold-with-boundary Q. We suppose that K satisfies the following condition $(\star)$: $(K \cap \operatorname{Int} Q) = K - \partial Q$ is connected and is an n-dimensional open submanifold of K. $\overline{K - \partial Q}$ is transverse to ∂Q and is an n-dimensional compact submanifold of Q. $K \cap \partial Q$ is a (not necessarily connected) n-dimensional compact submanifold of ∂Q. See Figure 7.1 for an example.

If $K \cap \partial Q \neq \phi$, we define the tubular neighborhood $N(K)$ of K in Q as follows: Take the tubular neighborhood of $\overline{K - \partial Q}$ in Q, and say X. Take the tubular neighborhood of $\overline{K - X}$ in $\overline{\partial Q - X}$, and say Y. Let $N(K) - X$ (resp. $\overline{N(K) - X}$) be the total space

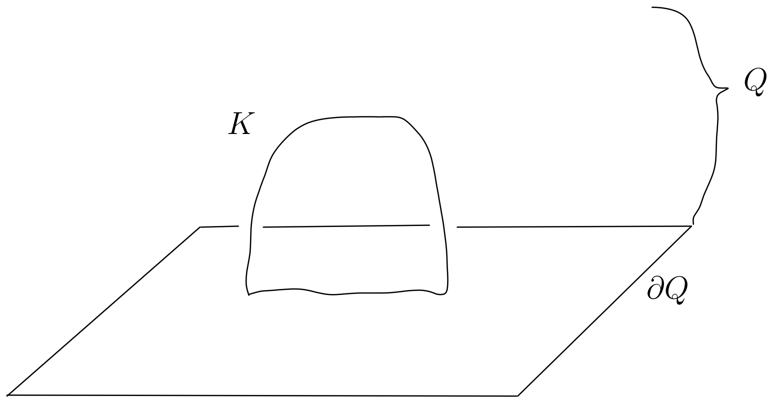


Figure 7.1: An example of K and Q.

of the restriction of 'the collar neighborhood of ∂Q in Q' to $Y - X$ (resp. Y) as a fiber bundle. Let $N(K)$ be diffeomorphic to $K \times D^2$.

Theorem 7.1. *Let $n \in \mathbb{N}$ and $n \geqq 3$. Let K be a closed oriented n-dimensional connected, $([\frac{n}{2}]-1)$-connected submanifold of Q. Let Q be S^{n+2}, B^{n+2}, or $S^{n+1} \times [0,1]$. We suppose that K satisfies the above condition $(\star)$. We do not suppose whether $K \cap \partial Q = \phi$ or $K \cap \partial Q \neq \phi$. Let $N(K)$ be the tubular neighborhood of K in Q. Then the following three conditions are equivalent:*

(i) $\pi_i(\overline{Q - N(K)}) = \begin{cases} \mathbb{Z} & \textit{if } i = 1 \\ 0 & \textit{if } 2 \leqq i \leqq [\frac{n-1}{2}]. \end{cases}$

(ii) *There is a $[\frac{n-1}{2}]$-connected Seifert hypersurface V for K.* (*Note: V is $(n+1)$-dimensional.*)

(iii) *There is a Seifert hypersurface V for K that has a special handle decomposition whose bottom is the empty set and whose top is ∂V.*

Proof of Theorem 7.1. Since $[K] = 0 \in H_n(Q;\mathbb{Z})$, there is a Seifert hypersurface V for K even if $K \cap \partial Q \neq \phi$. By using isotopy of V, we can suppose that Int $V \subset$ Int Q and that V is transverse to ∂Q. Let $N(V)$ be the tubular neighborhood of V in Q. Note that $\overline{\overline{Q - N(K)} - N(V)}$ is isotopic to $\overline{Q - N(V)}$ in Q.

We prove the following proposition.

Proposition 7.2. (iii) $\Longrightarrow$ (i).

Proof of Proposition 7.2. By van Kampen theorem $\pi_1(\overline{Q - N(V)}) \cong 1$. Reason: Note that $Q = N(V) \cup \overline{Q - N(V)}$ and that $N(V) \cap \overline{Q - N(V)}$ is a disjoint union of two copies of V. Take a 1-handle h^1 embedded in $\overline{Q - N(V)}$ and attach it to $N(V)$ so that the two attached parts are in different components. Use van Kampen theorem to a pair $(N(V) \cup h^1, \overline{Q - N(V)})$.

By Mayer-Vietoris exact sequence on $N(V)$ and $\overline{Q-N(V)}$, we have $H_i(\overline{Q-N(V)};\mathbb{Z}) \cong 0$ for $2 \leq i \leq [\frac{n-1}{2}]$.

Consider the infinite cyclic covering space $\widetilde{\overline{Q-N(K)}}$ of $\overline{Q-N(K)}$. Note that $\widetilde{\overline{Q-N(K)}}$ is a union of the lift of $N(V)$ and that of $\overline{Q-N(V)}$. By Mayor-Vietoris exact sequence on them, $H_i(\widetilde{\overline{Q-N(K)}};\mathbb{Z}) \cong 0$ if $1 \leq i \leq [\frac{n-1}{2}]$. By van Kampen theorem $\pi_1(\widetilde{\overline{Q-N(K)}}) \cong 1$. By Hurewicz theorem $\pi_i(\widetilde{\overline{Q-N(V)}}) \cong 0$ for $2 \leq i \leq [\frac{n-1}{2}]$.

By well-known facts on covering spaces, $\pi_i(\widetilde{\overline{Q-N(V)}}) \cong \pi_i(\overline{Q-N(V)})$ for $2 \leq i \leq [\frac{n-1}{2}]$. This completes the proof of Proposition 7.2. $\square$

Proposition 7.3. (i) $\Longrightarrow$ (ii).

Proof of Proposition 7.3. We prove the following claim.

Claim 7.4. *There is a simply-connected Seifert hypersurface for K.*

Proof of Claim 7.4. Take a Seifert hypersurface V for K. Let $\{g_1, ..., g_\mu\}$ be a set of generators of $\pi_1 V$. Let g_i also denote a circle that represents the element $g_i \in \pi_1 V$. Note that the dimension of V is $n+1$ and that $n+1 \geqq 4$. Hence we can suppose that g_i is embedded in V.

Since g_i is embedded in V, the intersection product of $[g_i] \in H_1(\overline{Q-N(K)};\mathbb{Z})$ and $[V, K] \in H_{n+1}(\overline{Q-N(K)}, \partial N(K);\mathbb{Z})$ is zero. Furthermore recall $\pi_1(\overline{Q-N(K)}) = \mathbb{Z}$. Hence we can take a continuous map $f_i : D_i^2 \to \overline{Q-N(K)}$ such that $f_i(\partial D_i^2) = g_i$.

Note that the dimension of $\overline{Q-N(K)}$ is $n+2$, and that $n+2 \geqq 5$. Hence we can suppose that f_i is an embedding map.

We need the following lemma.

Lemma 7.5. *Let C be an embedded circle in V. Suppose that there is an embedded 2-disc D in $\overline{Q-N(K)}$ such that $\partial D = C$. Note that D may intersect V. Let $N(C)$ be the tubular neighborhood of C in $\overline{Q-N(K)}$. Then we can suppose that $D \cap V \cap N(C)$ is C. (Recall that $n+2 \geqq 5$.)*

Proof of Lemma 7.5. Since V is orientable, $N(C) = C \times D^{n+1}$. Hence $\partial N(C)$ is the trivial S^n-bundle over C. Since $n \geq 2$, all sections of this trivial S^n-bundle over C are homotopic. There is a section perpendicular to V at C. Another section is defined by $D \cap N(C)$. Both sections are homotopic. Hence $D \cap V \cap N(C) = C$. This completes the proof of Lemma 7.5. $\square$

Let $N(g_i)$ be the tubular neighborhood of g_i in $\overline{Q-N(K)}$. By Lemma 7.5 $f_i(D_i^2) \cap V \cap N(g_i) = g_i$. Hence we can suppose that $f_i(D_i^2)$ intersects V transversely, that $(f_i(D_i^2) \cap V) - g_i$ is a disjoint union of some circles, and that f^{-1}(the circles) is in

the interior of D_i^2. Take an innermost circle of the circles $f_i(D_i^2) \cap V \subset D_i^2$. It bounds a disc in D_i^2. Note that (the disc) $\cap$ (the other circles)$= \phi$. Note that f(the disc) is embedded in $\overline{Q - N(K)}$ and that f(the disc) $\cap V = f$(the innermost circle). Take an embedded $(n+2)$-dimensional 2-handle whose core is f(the disc) and whose attaching part is embedded in V. Carry out a surgery on V by using this 2-handle. (Note that we remove the interior of $g_i \times D^n$ and add $D^2 \times S^{n-1}$.) Repeating this procedure on such circles in each D_i, we obtain a new V. By van Kampen theorem this new V is simply-connected. This completes the proof of Claim 7.4. □

It is trivial that Proposition 7.3 follows from the following Claims 7.6 and 7.7.

Claim 7.6. *Let $N(V)$ be the tubular neighborhood of V in $\overline{Q - N(K)}$. Note that $N(V) = V \times [-1, 1]$. Let $r \leqq [\frac{n-1}{2}]$. Suppose that there is an $(r-1)$-connected Seifert hypersurface. Then there is an $(r-1)$-connected Seifert hypersurface V with the following condition: For $t = 1, -1$, the homomorphism $\iota : \pi_r(V \times \{t\}) \to \pi_r(\overline{\overline{Q - N(K)} - N(V)})$ that is induced by the natural inclusion map is injective.*

Proof of Claim 7.6. The $r = 1$ case follows from Claim 7.4.

We prove the $r \geqq 2$ case. Take an $(r-1)$-connected Seifert hypersurface V for K. Suppose that $\alpha \in \pi_r(V \times \{t\})$ satisfies the condition $\iota(\alpha) = 0$ $(t \in \{1, -1\})$.

Note that the dimension of V is $n+1$, and that $2r \leqq n+1$. Hence α is represented by an embedded r-sphere in V.

Let α also denote this r-sphere. Then there is a continuous map $f : D^{r+1} \to \overline{Q - N(K)}$ such that $f(\partial D^{r+1}) = \alpha$. Since $\iota(\alpha) = 0$, $f(\text{Int}D^{r+1}) \cap V = \phi$.

Note that the dimension of $\overline{Q - N(K)}$ is $n+2$, that the dimension of D^{r+1} is $r+1$, that $r+1 \geqq 3$, and that $2(r+1) \leqq n+2$. Hence we can suppose that f is an embedding map.

Take an embedded $(n+2)$-dimensional $(r+1)$-handle whose core is $f(D^{r+1})$ and whose attaching part is embedded in V. Carry out a surgery on V by using this handle. (Note that we remove the interior of $S^r \times D^{n+1-r}$ from V and attach $D^{r+1} \times S^{n-r}$.) We obtain a new V. Repeating this procedure. Since $\pi_r V$ is finitely generated, $\iota : \pi_r(V \times \{t\}) \to \pi_r(\overline{\overline{Q - N(K)} - N(V)})$ becomes injective for $t = 1, -1$ after finite times of this procedure. This completes the proof of Claim 7.6. □

Claim 7.7. *Let $r \leqq [\frac{n-1}{2}]$. Let V be an $(r-1)$-connected Seifert hypersurface for K. Suppose that $\pi_r(V \times \{t\}) \to \pi_r(\overline{\overline{Q - N(K)} - N(V)})$ is injective for $t = 1, -1$. Then $\pi_r V = 0$.*

Proof of Claim 7.7. Take any element $\alpha \in \pi_r V$. Note that the dimension of V is $n+1$ and that $2r \leqq n+1$. Hence α is represented by an embedded r-sphere in V.

Let α also denote this r-sphere. Since $\pi_r(\overline{Q - N(K)}) = 0$, there is a continuous map $f : D^{r+1} \to \overline{Q - N(K)}$ such that $f(\partial D^{r+1}) = \alpha$.

Note that the dimension of $\overline{Q - N(K)}$ is $n + 2$, that the dimension of D^{r+1} is $r + 1$, that $r + 1 \geqq 3$, and that $2(r + 1) \leqq n + 2$. Hence we can suppose that f is an embedding map.

We prove the following:

Claim 7.8. *Let $N(\alpha)$ be the tubular neighborhood of α in $\overline{Q - N(K)}$. Then $f(D^{p+1}) \cap V \cap N(\alpha) = \alpha$.*

Proof of Claim 7.8. We need the following claim:

Claim 7.9. *Let $p \in \mathbb{N}$. Let $q \geqq p + 2$. Let α be an $\mathbb{R}^q$-bundle over S^p. Let τ be the tangent bundle of S^p. Let ε^r be the trivial $\mathbb{R}^r$-bundle over S^p. If $\alpha \oplus \tau = \varepsilon^{p+q}$, then $\alpha = \varepsilon^q$.*

Proof of Claim 7.9. Since $q \geqq p$, we have $\alpha = \varepsilon^1 \oplus \beta$, where β is an $\mathbb{R}^{q-1}$-bundle. Since $\tau \oplus \varepsilon^1 = \varepsilon^{p+1}$, $\alpha \oplus \tau = (\beta \oplus \varepsilon^1) \oplus \tau = (\tau \oplus \varepsilon^1) \oplus \beta = \varepsilon^{p+1} \oplus \beta$. Hence $\varepsilon^{p+q} = \varepsilon^{p+1} \oplus \beta$.

Recall that $\pi_i SO(n) \cong \pi_i(SO(n + 1))$ if $1 \leqq i \leqq n - 2$ and $n \in \mathbb{N} - \{1\}$: Reason; The exact sequence $\pi_i SO(n) \to \pi_i(SO(n + 1)) \to \pi_i S^n$.

Recall that $\mathbb{R}^r$-bundles over S^p are classified by $\pi_{p-1} SO(r)$. Note that $p-1 \leqq (q-1)-2$ and that $\varepsilon^{p+1} \oplus \beta$ is the trivial $\mathbb{R}^{p+q}$-bundle. Hence β is the trivial $\mathbb{R}^{q-1}$-bundle over S^p.

Hence $\beta \oplus \varepsilon^1$ is the trivial $\mathbb{R}^q$-bundle over S^p. This completes the proof of Claim 7.9. □

Let $N'(\alpha)$ be the tubular neighborhood of α in V (Recall that $N(\alpha)$ is the tubular neighborhood of α in $\overline{Q - N(K)}$.) By Claim 7.9, $N'(\alpha) = S^r \times D^{n+1-r}$ and $N(\alpha) = S^r \times D^{n+2-r}$. Hence $\partial N(\alpha) = S^r \times S^{n+1-r}$ is the trivial S^{n+1-r}-bundle over S^r. Since $n + 2 - r \geqq r + 2$, all sections of this trivial S^{n+1-r}-bundle over S^r are homotopic. There is a section that is perpendicular to $N'(V)$ at α. Another section is defined by $f(D^{r+1}) \cap N(\alpha)$. Both sections are homotopic. This completes the proof of Claim 7.8. □

Recall that f is an embedding map. By Claim 7.8 we can suppose that $f(D^{r+1})$ intersects V transversely and that $f(D^{r+1}) \cap V$ is a disjoint union of connected, closed, oriented, r-dimensional manifolds. Note that each of these r-manifolds is not an r-sphere in general. Take an innermost connected r-manifold M of these r-manifolds. Note that $f^{-1}(M)$ in D^{r+1} is diffeomorphic to M. There is an $(r + 1)$-dimensional compact connected, oriented, manifold W embedded in D^{r+1} such that $M = \partial W$. By the existence of W, M is a vanishing r-cycle in $\overline{\overline{Q - N(K)} - N(V)}$.

By Hurewicz theorem $\pi_r V = H_r(V; \mathbb{Z})$. By Hurewicz theorem, Mayor-Vietoris theorem, and van Kampen theorem, $\pi_1(\overline{\overline{Q - N(K)} - N(V)}) = 1$,

$\pi_i(\overline{\overline{Q - N(K)} - N(V)}) = 0$ for $2 \leqq i \leqq r-1$, and
$\pi_r(\overline{\overline{Q - N(K)} - N(V)}) = H_r(\overline{\overline{Q - N(K)} - N(V)}; \mathbb{Z})$. Hence there is an r-sphere embedded in V that is homologous to M and the homotopy class $[M] \in \pi_r(\overline{\overline{Q - N(K)} - N(V)})$ is zero. Since $\pi_r V \to \pi_r(\overline{\overline{Q - N(K)} - N(V)})$ is injective, the homotopy class $[M] \in \pi_r V$ is zero. By obstruction theory there is a continuous map $\overline{f} : W \to V$ such that $\overline{f}|_M = f|_M : M \to V$. Hence we remove M from $f(D^{p+1} \cap V)$ and keep the other connected manifolds than M by using a homotopy.

Repeating this procedure, we obtain a new f such that $f(D^{p+1} \cap V) = \phi$. Hence α is null-homotopic in $\overline{\overline{Q - N(K)} - N(V)}$. Since $\pi_r V \to \pi_r(\overline{\overline{Q - N(K)} - N(V)})$ is injective, α is null-homotopic in V. Hence $\pi_r V = 0$. This completes the proof of Claim 7.7. $\square$

This completes the proof of Proposition 7.3. $\square$

Proposition 7.10. (ii) $\Longrightarrow$ (iii) *if* $n \geqq 5$.

Proof of Proposition 7.10. We prove the following proposition that is stronger than Proposition 7.10.

Proposition 7.11. *Let* $n \geqq 5$. *There is a* $[\frac{n-1}{2}]$*-connected Seifert hypersurface* V *for* K. *Let* $B \cup T = K$, *where* B *(resp.* T*) may be the empty set. Let* B *and* T *be connected,* $([\frac{n}{2}] - 1)$*-connected, compact. Then there is a special handle decomposition of* V *whose bottom is* B *and whose top is* T.

Proof of Proposition 7.11. V satisfies the following condition $(*)$:
There is a handle decomposition of V with the following properties;

(i) The bottom is B. The top is T.

(ii) It has only one (resp. no) 0-handle if $B = \phi$ (resp. if $B \neq \phi$).

(iii) It has only one (resp. no) $(n+1)$-handle if $T = \phi$ (resp. if $T \neq \phi$).

Reason: Use 1-handles and cancel one or some handles if necessary.

Note that V is simply-connected and $(n+1)$-dimensional, that B is connected, and that $n + 1 \geqq 6$. Hence we have the following (see e.g. Lemma 1.21 in §1.3 of [19]).

Claim 7.12. *There is a Seifert hypersurface* V *for the* n*-dimensional submanifold* K *whose handle decomposition satisfies the above* $(*)$ *and has no 1-handle.*

Claim 7.13. *There is a Seifert hypersurface* V *for the* n*-dimensional submanifold* K *whose handle decomposition satisfies Claim 7.12 and that has no 2-handle.*

Proof of Claim 7.13. Take a handle decomposition of V that satisfies Claim 7.12. Take a sub-handle-decomposition

$$T_H = \begin{cases} (\text{the only one 0-handle})\cup(\text{all 2-handles}) & \text{if } B = \phi \\ (B \times [0,1])\cup(\text{all 2-handles}) & \text{if } B \neq \phi \end{cases}$$

of this handle decomposition.

Note that V is parallelizable, that B is simply-connected, that the dimension of B is n, and that $n \geqq 5$. Hence we have the following condition ($\sharp$)

$$T_H = \begin{cases} \natural^{\nu}(S^2 \times D^{n-1}) & \text{if } B = \phi \\ (B \times [0,1]\natural^{\nu}(S^2 \times D^{n-1})) & \text{if } B \neq \phi, \end{cases}$$

where $\nu \in \{0\} \cup \mathbb{N}$.

Note that V is connected, $[\frac{n-1}{2}]$-connected, compact, and that B is connected, $([\frac{n}{2}]-1)$-connected, compact. Hence $H_2(V, B; \mathbb{Z}) = \pi_2(V, B) = 0$ by using Mayor-Vietoris theorem, the homotopy exact sequence of pair, Hurewicz theorem.

By these facts and the above ($\sharp$) we can eliminate all 2-handles. This completes the proof of Claim 7.13. □

Note. Suppose that two compact connected manifolds A^a intersect B^b transversely in a simply-connected, connected, compact manifold C^{a+b+1}. Whitney trick does not work in general when a or b is 2 even if $a + b \geqq 5$. Reason: Let a (resp. b) be two. Whitney disc may intersect B (resp. A).

Claim 7.14. *There is a Seifert hypersurface V for the n-dimensional submanifold K whose handle decomposition satisfies Claim 7.13 and has no i-handle* $(1 \leqq i \leqq [\frac{n-1}{2}])$.

Proof of Claim 7.14. Note that V is connected, $[\frac{n-1}{2}]$-connected, compact, and that B is connected, $([\frac{n}{2}] - 1)$-connected, compact. Hence $H_i(V, B; \mathbb{Z}) = \pi_i(V, B) = 0$ $(1 \leqq i \leqq [\frac{n-1}{2}])$ by using Mayor-Vietoris theorem, the homotopy exact sequence of pair, Hurewicz theorem. By this fact and $n + 1 \geqq 6$, we can eliminate all i-handles $(1 \leqq i \leqq [\frac{n-1}{2}])$ by using Whitney trick. This completes the proof of Claim 7.14. □

Take the dual handle decomposition of the handle decomposition that satisfies Claim 7.14. Eliminate one or some handles if necessary in the same manner as above. This completes the proof of Proposition 7.11. □

This completes the proof of Proposition 7.10. □

Proposition 7.15. (ii) $\Longrightarrow$ (iii) *if* $n = 3, 4$.

Proof of Proposition 7.15. It is trivial that Proposition 7.15 follows from the following proposition.

Proposition 7.16. *The $n = 3, 4$ case of Proposition 7.11 holds.*

Proof of Proposition 7.16. Take a simply-connected Seifert hypersurface V for $K \subset S^{n+2}$. Since V is connected, compact, and B is connected, V satisfies the condition $(*)$ in the first paragraph of Proof of Proposition 7.11. Take a 1-handle h^1 of the handle decomposition. Since V is oriented, $B \cup h^1 = B \natural (S^1 \times D^n)$. Since $\pi_1 V = 1$, there is a continuous map $f : D^2 \to V$ such that $f(\partial D^2) = S^1 \times \{0\}$. Push off $f(\mathrm{Int} D^2)$ from V, keeping $f(\partial D^2)$, in the positive direction of the normal bundle of V in $\overline{S^{n+2} - N(K)}$. Thus we obtain a continuous map $g : D^2 \to S^{n+2} - N(K)$ such that $D^2 \cap V = \partial D^2 = S^1 \times \{0\}$.

Note that the dimension of $\overline{S^{n+2} - N(K)}$ is $n+2$, that the dimension of D^2 is 2, and that $n + 2 \geqq 2 \times 2$. Hence we can suppose that g is an embedding map.

Take an embedded $(n+2)$-dimensional 2-handle whose core is D^2 and whose attaching part is embedded in V. Carry out a surgery on V by using this 2-handle. (Note $n-1 \neq 1$. Note that we remove the interior of $S^1 \times D^n$ from V and add $D^2 \times S^{n-1}$.) Then the 1-handle h^1 is eliminated and a new $(n-1)$-handle is obtained. (Note that the dual of an $(n-1)$-handle is a 2-handle not a 1-handle.) We eliminate the 1-handle h^1 and obtain a new V. Repeating this procedure, we eliminate all 1-handles.

Take its dual handle decomposition. Since V is connected, compact, and B is connected, it also satisfies the condition $(*)$ (Of course we replace B (resp. T) in the condition $(*)$ with T (resp. B)). It has no n-handle. We can eliminate all 1-handles as above so that we do not obtain a new n-handle.

Therefore the new V has a handle decomposition such that the bottom is B, that the top is T, that it satisfies the condition $(*)$, and that it has no 1-handle, no n-handle. This completes the proof of Proposition 7.16. □

This completes the proof of Proposition 7.15. □

This completes the proof of Theorem 7.1. □

8. Proof of Theorem 6.1

We prove (ii)$\Longrightarrow$(i): Attach an embedded $(2p+2)$-dimensional $(p+1)$-handle to a simple Seifert hypersurface for K, carry out a surgery by using this handle, and obtain a new one. Then a simple Seifert matrix associated with the old one is $(-1)^p$-S-equivalent to that associated with the new one. Repeat this procedure.

Claim 8.1. (i) $\Longrightarrow$ (ii).

Proof of Claim 8.1. Let (X, Y) denote a pair (A manifold, its submanifold). Let $(X, Y) \times [0, 1]$ denote a pair (The manifold$\times[0, 1]$, its submanifold$\times[0, 1]$) which is a level preserving embedding. Take $(S^{2p+3}, K) \times [0, 1]$. Let V (resp. V') be a simple Seifert surface for K whose simple Seifert matrix is P (resp. P'). Take a $(2p+2)$-dimensional

submanifold $V \cup (K \times [0,1]) \cup V'$ in a $(2p+4)$-dimensional manifold $S^{2p+3} \times [0,1]$. By Proposition 7.11 and 7.16 we can suppose that a $(2p+3)$-dimensional Seifert hypersurface W for the $(2p+2)$-dimensional submanifold $V \cup (K \times [0,1]) \cup V'$ has a special handle decomposition
$(V \times [0,1]) \cup ((p+1)\text{-handles}) \cup ((p+2)\text{-handles}) \cup (V' \times [0,1])$. Let $\Phi : W \to [0,1]$ be a height function and a Morse function which gives this handle decomposition. By using isotopy we can suppose the following: Let $0 = t_0 \leqq t_1 ... \leqq t_\nu = 1$ be a partition of I satisfying

(i) Each t_i is a regular value of Φ.

(ii) At most one critical value of Φ lies in each interval (t_i, t_{i+1}).

Therefore it suffices to prove the case where Φ has only one critical point. If we change $[0,1]$ to $[0,-1]$, then the index ξ of critical point becomes $2p+3-\xi$. Hence it suffices to prove the case where Φ has only one critical point of index $(p+1)$. That is, it suffices to prove the case where W has a handle decomposition
$(V \times [0,1]) \cup (\text{only one } (p+1)\text{-handle}) \cup (V' \times [0,1])$. We can suppose that this only one $(p+1)$-handle is attached to V in $S^{2p+3} \times \{t\}$ for a t by a famous method in 'Morse theory with handle bodies'. Hence a simple Seifert matrix for V is $(-1)^p$-S-equivalent to that for V'. This completes the proof of Claim 8.1. □

This completes the proof of Theorem 6.1. □

9. Proof of Proposition 6.2

We show an example. Embed $S^3 \times D^3$ in S^7 such that $S^3 \times \{\text{the center}\}$ is the standard 3-sphere trivially embedded in S^7. Let the submanifold $\partial(S^3 \times D^3)$ in S^7 satisfy that the $S^3 \times D^3$ is its simple Seifert hypersurface whose simple Seifert matrix is a 1×1-matrix (0). This 5-dimensional submanifold is called K. We can suppose that $D^4 \times S^2$ bounds K. Hence ϕ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are 3-Seifert matrices for K, where ϕ is the empty matrix. Note that (0) is not S-equivalent to ϕ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We show another example. There is a closed oriented 3-dimensional submanifold K of S^5 with the following properties. (See §10 of [12] for this submanifold for detail.)

(1) There is a Seifert hypersurface V for K whose framed link representation is the $(2, 2a)$ torus link such that the framing of each component is zero ($a \in \mathbb{N} - \{1\}$). $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ is a simple Seifert matrix for K.

(2) There is a Seifert hypersurface W for K whose framed link representation is the $(2, 2a)$ torus link such that the framing of one component is zero and the other component is the dot circle. (Carry out a surgery on V by using a 5-dimensional 3-handle embedded in S^5 and obtain W.) W is a 4-dimensional homology ball. Hence ϕ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are Seifert matrices for K. Note that $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, where $a \in \mathbb{N} - \{1\}$, is not (-1)-S-equivalent to ϕ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. □

10. Proof of Theorem 6.3

First we prove Theorem 6.3.(1). Theorem 6.1 implies that (i) $\Longrightarrow$ (ii) in (1).

Lemma 10.1. (ii) $\Longrightarrow$ (i) *in* (1).

Proof of Proposition 10.1. It suffices to prove the case where $P_J = P_K$. By Theorem 6.1 and 'Theorem 7.1 and its proof', we can suppose that there is a simple Seifert hypersurface V_* with a special handle decomposition (the top ∂V_*, the bottom ϕ) whose simple Seifert matrix is P_* $(* = J, K)$. By Proposition 5.2, $P_* + (-1)^p\, {}^tP_*(* = J, K)$ is the intersection product on $H_p(V_*; \mathbb{Z})$.

Embed $(2p + 2)$-dimensional 0-handle h^0 in S^{2p+3} trivially. We can take p-spheres in ∂h^0 and attach embedded $(2p+2)$-dimensional $(p+1)$-handles to h^0 along the p-spheres so that a simple Seifert matrix associated with the result $h^0 \cup ((p + 1)$-handles) is P_* $(* = J, K)$.

Then V_J is diffeomorphic to V_K. Reason: The core of the attached part of each $(2p + 2)$-dimensional $(p + 1)$-handle is a p-sphere. The boundary of h^0 is a $(2p + 1)$-sphere. Furthermore $p \geqq 2$. By [5, 6, 34, 35, 36], the embedding type of an ordered disjoint union of p-spheres in ∂h^0 is determined by the set of the linking numbers of each pair of p-spheres.

The core of each $(2p + 2)$-dimensional $(p + 1)$-handle is $(p + 1)$-dimensional. S^{2p+3} is $(2p + 3)$-dimensional. Furthermore $p \geqq 2$. By [5, 6, 34, 35, 36], the embedding type of an ordered disjoint union of $(p+1)$-handles in $S^{2p+3}-\mathrm{Int}h^0$ keeping the attached part of each $(p + 1)$-handle is determined by the Seifert matrix.

Therefore there is a diffeomorphism map $f : S^{2p+3} \to S^{2p+3}$ such that $f(V_J) = V_K$.

This completes the proof of Proposition 10.1. □

Next we prove Theorem 6.3.(2). (i) $\Longrightarrow$ (ii) in (2) is trivial.

Lemma 10.2. (ii) $\Longrightarrow$ (i) *in* (2).

Proof of Proposition 10.2. By [5, 6, 34, 35, 36], the embedding type of an ordered disjoint union of 2-handles in $S^5 - \mathrm{Int} h^0$ keeping the attached part of each 3-handle is determined by the Seifert matrix.

Therefore V_J is equivalent to V_K as submanifolds in S^5. This completes the proof of Proposition 10.2. □

This completes the proof of Theorem 6.3. □

11. Proof of Theorem 4.3

Claim 11.1. *There is a simple Seifert matrix P_* for $*$ ($* = J, K$) with the following property: Each element of P_J is the same as that of P_K except for only one diagonal element. They differ by one.*

Proof of Claim 11.1. There is a $(2p+3)$-ball B where the twist-move is carried out. Take a $(2p+2)$-dimensional $(p+1)$-handle h associated with the twist move that is embedded in B. Let Z be a $(2p+1)$-dimensional closed oriented submanifold $(J - B) \cup (h \cap \partial B) \subset S^{2p+3}$. Thus (J, K, Z) is a twist-move-triple. Note that Z is embedded in $\overline{S^{2p+3} - B}$. By the construction of Z, Z is $(p-1)$-connected. Furthermore $\overline{S^{2p+3} - N(J)}$ (resp. $\overline{S^{2p+3} - N(K)}$) is made by attaching one $(2p+3)$-dimensional $(p+2)$-handle to $\overline{\overline{S^{2p+3} - B} - N(Z)}$. Hence $\pi_i(\overline{S^{2p+3} - N(J)}) = \pi_i(\overline{S^{2p+3} - N(K)})$ $= \pi_i(\overline{\overline{S^{2p+3} - B} - N(Z)})$ $(1 \leqq i \leqq p)$.

By the assumption, $J = A \otimes [2]$ holds and A is a $(2p-1)$-dimensional connected, $(p-2)$-connected, simple submanifold $\subset S^{2p+1}$. Hence, by [8, 11], J is a $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold $\subset S^{2p+3}$.

Hence Z is a $(2p+1)$-dimensional connected, $(p-1)$-connected, simple submanifold of $\overline{S^{2p+3} - B}$. By Theorem 7.1 there is a p-connected Seifert hypersurface W in $\overline{S^{2p+3} - B}$ for Z that has a special handle decomposition (the top Z, the bottom ϕ). Note that the handle h is attached to W. By using this $W \cup h$ we obtain a simple Seifert hypersurface V_J (resp. V_K) for J (resp. K) that has a special handle decomposition. This completes the proof of Claim 11.1. □

Claim 11.2. $(-1)^{p-1} P_J$ *is a simple Seifert matrix for A.*

Proof of Claim 11.2. Let P_A be a simple Seifert matrix for A.$\cdots$(1)
We prove the following Claims 11.3 and 11.4.

Claim 11.3. $(-1)^{p-1} P_A$ *is* $(-1)^p$*-S-equivalent to* P_J.

Proof of Claim 11.3. Since $A \otimes [2] = J$, $(-1)^{p-1} P_A$ is a simple Seifert matrix for J. (Reason: Note 6.10. Note that a Seifert matrix for [2] is a 1×1-matrix $\Lambda_2 = (1)$. Recall that Λ_2 is defined right before Theorem 5.3.) By Theorem 6.3, we have Claim 11.3. □

Claim 11.4. *$(-1)^{p-1}P_A$ is $(-1)^{p+1}$-S-equivalent to P_J.*

Proof of Claim 11.4. By Note 6.10, $(-1)^p P_J$ is a simple Seifert matrix for $J \otimes [2]$. By Note 6.10 and Claim 11.3, $(-1)^p(-1)^{p-1}P_A$ is a simple Seifert matrix for $J \otimes [2]$. By Theorem 6.3 it holds that $(-1)^p(-1)^{p-1}P_A$ is $(-1)^{p+1}$-S-equivalent to $(-1)^p P_J$. Hence we have Claim 11.4. $\square$

Note. Claims 11.3 and 11.4 hold on time. Reason: Recall Proposition 5.2. The way of making the intersection matrix from a pair of related positive Seifert matrix and negative one depends on whether p is odd or even in general.

By Claim 11.4 P_A is $(-1)^{p+1}$-S-equivalent to $(-1)^{p-1}P_J$. This fact, the above (1), and Theorem 6.1 imply Claim 11.2. $\square$

By Claims 11.1 and 11.2, we can make a $(2p-1)$-dimensional connected, $(p-1)$-connected, simple submanifold B of S^{2p+1} from A by one twist move so that a simple Seifert matrix of B is $(-1)^{p-1}P_K$. By [8, 11] and Theorem 6.3, we have (i) (ii) (iii) in Theorem 4.3. $\square$

12. Proof of Theorem 4.1

Claim 12.1. *There is a simple Seifert matrix P_* for $*$ $(* = J, K)$ with the following property: Let $p_{*,ij}$ be the (i,j)-element of P_*. Let P_* be a $c \times c$-matrix $(c \in \mathbb{N})$. There are natural numbers $a, b \leqq c$ such that $a \neq b$ and that*

$$\begin{cases} p_{J,ij} = p_{K,ij} - 1 & \text{if } (i,j) = (a,b) \\ p_{J,ij} = p_{K,ij} & \text{if } (i,j) \neq (a,b) \text{ and if } (i,j) \neq (b,a). \end{cases}$$

(Recall that the (b,a)-element is the same as the (a,b)-element.)

Proof of Claim 12.1. There is a $(4\mu+3)$-ball B where the $(2\mu+1, 2\mu+1)$-pass-move is carried out. Take $(4\mu+2)$-dimensional $(2\mu+1)$-handles h and h' associated with the $(2\mu+1, 2\mu+1)$-pass-move that are embedded in B. Let Z be a $(4\mu+1)$-dimensional closed oriented submanifold $(J - B) \cup (h \cap \partial B) \cup (h' \cap \partial B) \subset S^{4\mu+3}$. Thus (J, K, Z) is a $(2\mu+1, 2\mu+1)$-pass-move-triple. Note that Z is embedded in $\overline{S^{4\mu+3} - B}$. By the construction of Z, Z is $(2\mu-1)$-connected. Furthermore $\overline{S^{4\mu+3} - N(J)}$ (resp. $\overline{S^{4\mu+3} - N(K)}$) is made by attaching two $(4\mu+3)$-dimensional $(2\mu+2)$-handles and one $(4\mu+3)$-dimensional $(4\mu+2)$-handles to $\overline{\overline{S^{4\mu+3} - B} - N(Z)}$. Hence $\pi_i(\overline{S^{4\mu+3} - N(J)}) = \pi_i(\overline{S^{4\mu+3} - N(K)}) = \pi_i(\overline{\overline{S^{4\mu+3} - B} - N(Z)})$ $(1 \leqq i \leqq 2\mu)$.

By the assumption $J = A \otimes^\mu \text{Hopf}$ holds and A is a 1-link. By [8, 11] J is a $(4\mu+1)$-dimensional connected, $(2\mu-1)$-connected, simple submanifold $\subset S^{4\mu+3}$.

Hence Z is a $(4\mu+1)$-dimensional connected, $(2\mu-1)$-connected, simple submanifold of $\overline{S^{4\mu+3} - B}$. By Theorem 7.1 there is a 2μ-connected Seifert hypersurface W in $\overline{S^{4\mu+3} - B}$

for Z that has a special handle decomposition (the top Z, the bottom ϕ). Note that the handles h and h' are attached to W. By using this $W \cup h \cup h'$ we obtain a simple Seifert hypersurface V_J (resp. V_K) for J (resp. K) that has a special handle decomposition. Hence we have Claim 12.1. $\square$

Let P_A be a Seifert matrix for the 1-knot $A.\cdots(1)$

Claim 12.2. *P_A is S-equivalent to $(-1)^\mu P_J$.*

Proof of Claim 12.2. Since $J = A \otimes^\mu$Hopf, $(-1)^\mu P_A$ is a simple Seifert matrix for J. (Reason: See Note 6.10. Note that a Seifert matrix for the Hopf link is a 1×1-matrix $\Lambda_{2,2} = (-1)$. Recall that $\Lambda_{2,2}$ is defined right before Theorem 5.3.) By Theorem 6.3 $(-1)^\mu P_A$ is S-equivalent to P_J. Hence we have Claim 12.2. $\square$

By Claim 12.2 we have the following:

Claim 12.3. *There is a 1-link A' whose Seifert matrix is $(-1)^\mu P_J$.*

By Claim 12.1 and 12.3, we have the following:

Claim 12.4. *We can make a 1-link B from A' by one pass-move so that a Seifert matrix of B is $(-1)^\mu P_K$.*

A is pass-move equivalent to A' because of the above (1), Claims 12.2 and 12.3. By this and Claim 12.4 A is pass-move equivalent to B. By Theorem 6.3 and Claim 12.4 $B \otimes^\mu$Hopf is equivalent to K. This completes the proof of Theorem 4.1. $\square$

Note 12.5. Theorem 4.1 of [13], which is cited right above Theorem 4.2, is also proved by using Theorem 6.3. Outline of the proof: There is a Seifert surface V_A (resp. V_B) for A (resp. B) such that V_A is obtained from V_B by one pass-move. There is a Seifert matrix S_A (resp. S_B) associated with V_A (resp. V_B). By [11], $(-1)^\mu S_A$ (resp. $(-1)^\mu S_B$) is a $(2\mu+1)$-Seifert matrix for $K_A \otimes^\mu$Hopf (resp. $K_B \otimes^\mu$Hopf). By the definition of the $(2\mu+1, 2\mu+1)$-pass-move and Theorem 6.3, there is a simple Seifert hypersurface W_A (resp. W_B) for $A \otimes^\mu$Hopf (resp. $B \otimes^\mu$Hopf) such that $(-1)^\mu S_A$ (resp. $(-1)^\mu S_B$) is a $(2\mu+1)$-Seifert matrix associated with W_A (resp. W_B) and such that W_A is obtained from W_B by one $(2\mu+1, 2\mu+1)$-pass-move. Hence Theorem 4.1 of [13] holds.

Acknowledgement. The authors would like to thank Osamu Saeki for the valuable discussion.

References

[1] W. Browder: Surgery on simply-connected. manifolds, *Springer-Verlag Berlin Heidelberg New York.* (1972).

[2] S. E. Cappell and J. L. Shaneson: There exist inequivalent knots with the same complement. *Ann. of Math.* 103 (1976) 349353.

[3] A. H. Durfee: Fibered knots and algebraic singularities, *Topology* 13 (1974) 47-59.

[4] H. Gluck: The embedding of two-spheres in the four-sphere, *Transactions of the American Mathematical Society* 104 (1962) 308-333.

[5] A. Haefliger: Differentiable imbeddings, *Bull. Amer. Math. Soc.* 67 (1961) 109-112.

[6] A. Haefliger: Plongements diffrentiables de varits dans varits. *Commentarii mathematici Helvetici* 36 (1962) 47 - 82.

[7] M. Kato: A classification of simple spinnable structures on a 1-connected Alexander manifold, *J. Math. Soc. Japan* 26 (1974) 454-463.

[8] L. H. Kauffman: Products of knots, *Bulletin of the American mathematical society* 80 (1974) 1104-1107.

[9] L. H. Kauffman: On Knots, *Ann of math studies* 115 (1987).

[10] L. H. Kauffman: Super Twist Spinning, *Festschrift for Antonio Plans, Universita de Zaragoza Press* (1990) 139-154.

[11] L. H. Kauffman and W. D. Neumann: Products of knots, branched fibrations and sums of singularities, *Topology* 16 (1977) 369-393.

[12] L. H. Kauffman and E. Ogasa: Local moves of knots and products of knots, *Volume three of Knots in Poland III, Banach Center Publications* 103 (2014), 159-209, arXiv: 1210.4667 [math.GT].

[13] L. H. Kauffman and E. Ogasa: Local moves of knots and products of knots II, arXiv: 1406.5573[math.GT].

[14] R. C. Kirby: The topology of 4-manifolds *Springer-Verlag, Berlin, New York* (1989).

[15] J. Levine: Unknotting spheres in codimension two, *Topology* 4 (1965) 9-16.

[16] J. Levine: Polynomial invariants of knots of codimension two. *Ann. of Math. 84, 537–554*, (1966).

[17] J. Levine: Knot cobordism in codimension two, *Comment. Math. Helv.* 44 (1969) 229-244.

[18] J. Levine: An algebraic classification of some knots of codimension two. *Comment. Math. Helv.* 45 (1970) 185–198.

[19] W. Lück: A basic introduction to surgery theory. *ICTP Lecture Notes Series 9, Band 1, of the school "High-dimensional manifold theory" in Trieste, May/June (2001), Abdus Salam International Centre for Theoretical Physics, Trieste.*

[20] J. Milnor: Singular Points of Complex Hypersurfaces. Princeton Univ. Press (1969).

[21] E. Ogasa: Intersectional pairs of n-knots, local moves of n-knots and invariants of n-knots, *Mathematical Research Letters* 5 (1998) 577-582, Univ. of Tokyo preprint UTMS 95-50.

[22] E. Ogasa: The intersection of spheres in a sphere and a new geometric meaning of the Arf invariants, *Journal of knot theory and its ramifications* 11 (2002) 1211-1231, Univ. of Tokyo preprint series UTMS 95-7, arXiv: 0003089 [math.GT].

[23] E. Ogasa: Ribbon-moves of 2-links preserve the μ-invariant of 2-links, *Journal of knot theory and its ramifications* 13 (2004) 669–687, UTMS 97-35, arXiv: 0004008 [math.GT].

[24] E. Ogasa: Ribbon-moves of 2-knots: The Farber-Levine pairing and the Atiyah-Patodi-Singer-Casson-Gordon-Ruberman $\tilde{\eta}$ invariant of 2-knots, *Journal of Knot Theory and Its Ramifications* 16 (2007) 523-543, arXiv: math.GT/0004007, UTMS 00-22, arXiv: 0407164 [math.GT].

[25] E. Ogasa: Local move identities for the Alexander polynomials of high-dimensional knots and inertia groups, *Journal of Knot Theory and Its Ramifications* 18 (2009) 531-545, UTMS 97-63. arXiv: 0512168 [math.GT].

[26] E. Ogasa: A new obstruction for ribbon-moves of 2-knots: 2-knots fibred by the punctured 3-torus and 2-knots bounded by the Poicaré sphere, arXiv: 1003.2473 [math.GT].

[27] E. Ogasa: An introduction to high dimensional knots, arXiv: 1304.6053 [math.GT].

[28] O. Saeki: Knotted homology 3 -spheres in S 5, *J. Math. Soc. Japan* 40 (1988) 65-75.
[29] O. Saeki: Theory of Fibered 3-Knots in S5 and its Applications, *J. Math. Sci. Univ. Tokyo* 6 (1999) 691756.
[30] S. Smale: Generalized Poincaré conjecture in dimensions greater than four, *Annals of Math.* 74 (1961) 391–406.
[31] C. T. C. Wall: Diffeomorphisms of 4-manifolds *J. London Math. Soc.* 39 (1964), 131-140.
[32] C. T. C. Wall: On simply connected 4-manifolds *J. London Math. Soc.* 39 (1964), 141-149.
[33] C. T. C. Wall: Surgery on compact manifolds, *Academic Press, New York and London* (1970).
[34] H. Whitney: Differentiable manifolds, *Annals of Mathematics Second Series* 37 (1936) 645-680.
[35] H. Whitney: The Self-intersections of a smooth n-manifold in $2n$-space, *Annals of Mathematics Second Series* 45 (1944) 220-246.
[36] W. T. Wu: On the isotopy of Cr-manifolds of dimension n in Euclidean (2n+1)-space, *Sci. Record (N.S.)* 2 (1958) 271-275.

Louis H. Kauffman: Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, Illinois 60607-7045, USA kauffman@uic.edu

Eiji Ogasa: Computer Science, Meijigakuin University, Yokohama, Kanagawa, 244-8539, Japan pqr100pqr100@yahoo.co.jp ogasa@mail1.meijigkakuin.ac.jp